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Overturning of nonlinear acoustic waves. Part 2 Relaxing gas dynamics

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We consider finite-amplitude acoustic disturbances propagating through media in which relaxation mechanisms, such as those associated with the vibration of polyatomic molecules, are significant. While the effect of these relaxation modes is to inhibit the wave steepening associated with nonlinearity, whether a particular mode is sufficient to prevent the occurrence of multi-valued solutions will depend on the form of the disturbance and on the characteristic parameters of the relaxation. Analysis of this condition is necessary in order to reveal which physical mechanisms actually determine the evolution of the wave profile. This then dictates the scaling of any embedded shock regions. Sufficient conditions for the occurrence of multi-valued solutions are obtained analytically for periodic waves, hence proving that in certain circumstances relaxation is in fact insufficient in fully describing the wave propagation. A much more precise criterion is then obtained numerically. This uses the techniques described in Part 1 for analysing the phenomenon of wave overturning using intrinsic coordinates. Illustrations are provided of the development of a harmonic signal for different classes of material parameters.

1. Introduction

In Part 1 (Hammerton & Crighton 1993), a method of solving nonlinear wave equations using an intrinsic coordinate method was described. Using such a method, the existence, or otherwise, of multi-valued solutions can be determined. While such multi-valued solutions are meaningless in a physical context, their appearance does point to a breakdown in the model equation, and hence the increased importance of some physical mechanism previously ignored. In this paper we use these methods to investigate finite-amplitude acoustic propagation through relaxing media. The essential feature of a relaxing fluid is that the partition of energy among the available modes does not respond instantaneously to changes imposed by a time-dependent flow. Each physical relaxation mode has a characteristic timescale, and if the relaxation time associated with a particular physical process is comparable to the disturbance timescale, then the effect of relaxation must be accounted for in determining the evolution of the disturbance.

For a gaseous medium, it is the partition of internal vibration energy within polyatomic molecules that gives rise to significant relaxation effects. For air, relaxing modes associated with O_2 and N_2 are dominant, though both relaxation times are very sensitive to the presence of H_2O. Thus acoustic propagation through the atmosphere is affected by humidity (see, for example, Pierce 1981, p. 554). For acoustic propagation through seawater, relaxation processes are chemical in origin rather than physical. Pressure-dependent chemical relaxations can lead to absorption and dispersion of sound, and for salt water it has been found that $MgSO_4$ (magnesium sulphate) and $B(OH)_3$ (boric acid) are the main contributors to relaxation processes even though they are present only in extremely low concentrations. A discussion of relaxation effects in seawater, including numerical parameter values, is given in Fisher & Simmons (1977). In addition, relaxation effects arise in acoustic propagation through aerosols or dusty gases, where a solid phase of fine rigid particles exists within a gaseous phase. Two relaxation processes are present here: the adjustment of the particle velocity to the surrounding gas velocity, and the temperature adjustment of the particles to the gaseous surroundings. Though the origin of the relaxing mode is very different in each of the above cases, the effect on the propagation of a disturbance through such media is the same.

Compared to the bulk of literature available for finite-amplitude propagation when thermoviscous diffusion is the dominant attenuation mechanism, much less attention has been paid to media in which relaxation mechanisms are significant. The governing equations were derived independently by Polyakova, Soluyan & Khokhlov (1962), Blythe (1969) and Ockendon & Spence (1969) for media in which a single time characterizes the rate of change of all the non-equilibrium variables. However, analysis was essentially restricted to travelling wave solutions describing the propagation of a forward-facing step, the analogue of Taylor's solution for a diffusion-resisted shock. A condition for the existence of such solutions was obtained in terms of the shock amplitude and the material parameters. In cases where the amplitude is too large for the nonlinear wave steepening to be adequately balanced by relaxation alone, diffusion becomes locally significant with the formation of a thin viscous sub-shock, and we have a partly dispersed relaxation shock (Lighthill 1956). In this paper we consider the propagation of periodic disturbances and investigate the conditions for which relaxation mechanisms dominate throughout, when the solution to the nonlinear wave equation including only relaxation processes remains single-valued and the wave remains fully dispersed. For a sinusoidal wave, conditions are accurately determined in terms of the signal amplitude, the signal frequency and the material parameters. The situation envisaged here is of acoustic propagation away from a harmonically oscillating piston, after any transient behaviour associated with the initial starting motion of the piston has died away. The corresponding start-up problem is dealt with by Johannesen & Scott (1978), and is much simpler, because only conditions near the leading characteristic (with the infinite-frequency 'frozen' sound speed) need be considered. In the steady-state response considered in this paper, no such limitation can be justified, and the overturning problem is a global one.

In §2, the governing equation for finite-amplitude propagation through a relaxing medium is introduced. The standard travelling wave solutions for media with only one relaxation mode are reproduced to provide some insight into the effect of relaxation on periodic disturbances. In §3, conditions *sufficient* to ensure that wave overturning occurs are derived using functional analysis arguments. Hence it is proved that relaxation alone can be insufficient in balancing nonlinear steepening and that other physical processes must become significant, as opposed to merely proving that single-valued travelling wave solutions do not exist. In §4, a very much more precise criterion for wave overturning to occur is obtained numerically for sinusoidal signals, using the intrinsic coordinate approach detailed in Part 1.

2. Governing equation and travelling wave solutions

When considering the effect of relaxation processes on the propagation of a disturbance through a medium, the magnitude of each relaxation time relative to the characteristic disturbance timescale becomes an important parameter. Analysis of the linear propagation of a harmonic disturbance, with relaxation effects included, reveals that phase velocity for the signalling problem increases monotonically with signal frequency, from a_0 to a_∞. At low frequency, the effect of relaxation is small since the adjustment to thermal equilibrium is virtually instantaneous; hence a_0, the low-frequency sound speed, is also known as the equilibrium sound speed. At the other end of the scale, for a very high-frequency signal, the internal vibration energy of the particular molecular species can never adjust to the change in equilibrium and is *frozen*; hence a_∞ is known as the frozen sound speed. Attenuation per wavelength peaks at a signal frequency equal to the relaxation frequency. Lighthill (1956) gives general expressions for attenuation and phase-velocity due to relaxation.

With the inclusion of these linear relaxation effects in the derivation of the nonlinear wave equation, the evolution of a right-running disturbance, referred to a frame moving at the equilibrium sound speed, is given by the non-dimensional equation

$$u_t + uu_\theta = \Delta u_{\theta\theta} - \sum_\nu K_\nu \mathrm{e}^{\theta/\Omega_\nu} \int^\theta \mathrm{e}^{-\theta'/\Omega_\nu} u_{\theta'\theta'} \,\mathrm{d}\theta'. \tag{2.1}$$

Here $u = \hat{u}/U_0$, $\theta = (\hat{x} - a_0 \hat{t})\,\omega/a_0$, $t = \hat{t}\omega(\gamma+1)\,U_0/2a_0$, where U_0 is the maximum signal amplitude, ω a typical signal frequency, γ the adiabatic exponent or an equivalent parameter for condensed fluids, and $\hat{u}$, $\hat{x}$, $\hat{t}$ are the dimensional signal amplitude, spatial coordinate and time, respectively. Three dimensionless parameters arise:

$$\Delta = \frac{2\omega\delta}{a_0\,U_0(\gamma+1)}, \quad K_\nu = \frac{2(\Delta a)_\nu}{U_0(\gamma+1)}, \quad \Omega_\nu = \omega\tau_\nu, \tag{2.2}$$

where τ_ν is the relaxation time for species ν and $(\Delta a)_\nu$ the difference between frozen and equilibrium sound speeds due to species ν. It should be noted that Ω_ν depends only on the identity of the species ν, while K_ν also depends on its relative concentration. The model equation is valid only if the changes in wave shape occur over a large number of wavelengths, which requires that nonlinear effects are small ($U_0/a_0 \ll 1$), diffusion is small ($\omega\delta/a_0^2 \ll 1$), and finally that the energy in each relaxation mode is small ($(\Delta a)_\nu/a_0 \ll 1$). However, it can be seen that these conditions place no restriction on the magnitude of the parameters defined in (2.2).

The remainder of this paper is concerned with acoustic propagation through a medium characterized by a single relaxation time, when the non-dimensional model wave equation (2.1) can be written

$$\left(1 - \Omega\frac{\partial}{\partial\theta}\right)[u_t + uu_\theta - \Delta u_{\theta\theta}] = \Gamma u_{\theta\theta}, \tag{2.3}$$

where $\Gamma = K\Omega$. This is the form considered in the past by Blythe (1969) and Ockendon & Spence (1969); in both papers an exact travelling wave solution is obtained for the case $\Delta = 0$. Writing $u(\theta, t)$ as $V(\phi)$, where $\phi = \theta - \frac{1}{2}t$, V is given implicitly by

$$\Omega^{-1}(\phi + \phi_0) = (1+2K)\ln(1-V) + (1-2K)\ln V. \tag{2.4}$$

For $K > \frac{1}{2}$, $V(\phi)$ is smooth with $V \to 1$ as $\phi \to -\infty$ and $V \to 0$ as $\phi \to \infty$. Thus the solution describes a steadily translating pressure step, such as that due to a compressive piston moving at constant speed. In dimensional quantities this condition is equivalent to $U_0 < 4(\Delta a)/(\gamma+1)$, which is the well-established condition for a fully dispersed relaxing shock – see Lighthill (1956). At the critical value $K = \frac{1}{2}$, the gradient of the profile is discontinuous at the head of the shock, and the velocity of the shock front is the frozen sound speed a_∞. For $K < \frac{1}{2}$, the boundary condition ahead of the disturbance cannot be satisfied and no such travelling wave solution exists. In the latter case the inclusion of diffusivity ensures a single-valued solution, with the appearance of a sub-shock, the width of which scales with Δ. The remainder of this paper is concerned with the low diffusivity limit ($\Delta \ll 1$) for periodic disturbances. Conditions on Ω and Γ are determined for which the term $\Delta u_{\theta\theta}$ can be ignored throughout, in comparison to those cases in which viscosity becomes locally significant and introduces a new, shorter lengthscale into the wave profile. In other words, we seek the region of the (Ω, Γ)-plane in which the wave remains fully dispersed throughout its evolution.

3. Sufficient conditions for wave overturning

In this section, we prove that a periodic waveform propagating through a relaxing medium according to

$$\left(1-\Omega\frac{\partial}{\partial\theta}\right)[u_t+uu_\theta] = \Gamma u_{\theta\theta}, \tag{3.1}$$

will become multi-valued in finite time for certain values of the two relaxation parameters Γ and Ω. If $\Gamma = 0$, relaxation effects are absent and the inviscid Burgers equation is obtained, leading to overturning for any initial profile with portions of negative slope. However, here we determine conditions on the two parameters for finite rate relaxation which, if satisfied, still result in the occurrence of multi-valued solutions. Thus it is proved, analytically, that under certain conditions linear attenuation and dispersion due to relaxation effects alone are not sufficient to prevent nonlinear wave overturning. The method used to study this issue is an adaptation of one used by Hunter (1989) in the context of high-frequency water waves on a very shallow rotating fluid.

Here we restrict attention to an initially sinusoidal disturbance, $u_0(\theta) = \sin\theta$. An analysis for more general initial data is presented elsewhere (Hammerton 1990). Changing to a frame moving with the frozen sound speed, we have

$$u_t+uu_\phi = -\frac{\Gamma}{\Omega^2}\left(u+\frac{e^{\phi/\Omega}}{\Omega}\left[I(\phi)-\frac{1}{1-e^{-2\pi/\Omega}}I(2\pi)\right]\right) \equiv L(\phi,t), \tag{3.2}$$

where
$$I(\phi,t) = \int_0^\phi u(\phi',t)\,e^{-\phi'/\Omega}\,d\phi' \quad \text{and} \quad \phi = \theta-\frac{\Gamma}{\Omega}t. \tag{3.3}$$

Setting $u_m(t) = \sup|u(\phi,t)|$ we see that

$$\left|\frac{e^{\phi/\Omega}}{\Omega}\left[I(\phi)+\frac{I(2\pi)}{e^{-2\pi/\Omega}-1}\right]\right| \leqslant \frac{e^{\phi/\Omega}}{\Omega}\left[\left|I(\phi)-I(2\pi)\right|+\frac{e^{-2\pi/\Omega}}{1-e^{-2\pi/\Omega}}\left|I(2\pi)\right|\right]$$
$$\leqslant u_m, \tag{3.4}$$

and then it follows that $du_m/dt \leqslant 0$. Thus, the wave amplitude decreases monotonically with time, and so

$$|L(\phi,t)| \leqslant 2\Gamma/\Omega^2. \tag{3.5}$$

With the right-hand side of (3.2) bounded, the characteristic form of (3.2) can now be used to obtain conditions for which wave overturning will occur.

Using Z as the characteristic variable, and expressing all terms as functions of Z and t by

$$\phi = X(Z,t), \quad u = W(Z,t), \quad L = G(Z,t), \tag{3.6}$$

the governing equations obtained, together with initial conditions, are

$$\left.\begin{aligned} X_t &= W, \quad X(Z,0) = Z, \\ W_t &= G, \quad W(Z,0) = u_0(Z). \end{aligned}\right\} \tag{3.7}$$

The function $u(\phi, t)$ becoming a multi-valued function at some finite time t^* corresponds to the characteristic variable taking more than one value for some ϕ at that t^*. Since $X_Z(Z,0) = 1$, Z becomes a multi-valued function of ϕ if $X_Z(Z,t^*) < 0$ for some Z. The remainder of this analysis is concerned with finding a range of parameter values for which this certainly happens.

Introducing the notation

$$I_1(Z,t) = \int_0^t G_Z(Z,s)\,\mathrm{d}s \quad \text{and} \quad I_2(Z,t) = \int_0^t \int_0^s G_Z(Z,v)\,\mathrm{d}v\,\mathrm{d}s, \tag{3.8}$$

the integration of (3.7), and elimination of W, gives an expression for X_Z:

$$X_Z = 1 + tu_0' + I_2. \tag{3.9}$$

Thus if it can be proved that for some (Z,t), the solution for $G(Z,t)$ is such that $tu_0'(Z) + I_2(Z,t) < -1$, then it will have been shown that the wave profile has become multi-valued by this time. For the relaxation equation (3.1), $G(Z,t)$ satisfies

$$\frac{\partial}{\partial Z}(G\,\mathrm{e}^{-X/\Omega}) = -\frac{\Gamma}{\Omega^2} W_Z\,\mathrm{e}^{-X/\Omega}, \tag{3.10}$$

and hence a single equation governing the evolution of $G(Z,t)$ is obtained:

$$G_Z = \frac{G}{\Omega}(1 + tu_0' + I_2) - \frac{\Gamma}{\Omega^2}(u_0' + I_1). \tag{3.11}$$

Defining Z^* by $u_0'(Z^*) = -1$ and using the bound on $|G|$ provided by (3.5), it readily follows that

$$|G_Z(Z^*,t)| \leqslant \left(\frac{\Gamma t^2}{\Omega^3} + \frac{\Gamma t}{\Omega^2}\right) C(t) + \frac{2\Gamma}{\Omega^3}(t+1) + \frac{\Gamma}{\Omega^2}, \tag{3.12}$$

where

$$C(t) = \sup_{0 \leqslant s \leqslant t} |\hat{G}_Z(s)|. \tag{3.13}$$

The right-hand side of this inequality is a monotonically increasing function of t, so it follows that $C(t)$ also obeys this inequality. Thus $C(t)$ satisfies

$$\left(\frac{\Omega^2}{\Gamma} - t - \frac{t^2}{\Omega}\right) C(t) \leqslant \left(\frac{2}{\Omega} + 1\right) + \frac{2t}{\Omega}. \tag{3.14}$$

From (3.9) evaluated at $Z = Z^*$, it follows that

$$|X_Z(Z^*,t) + t - 1| \leqslant \tfrac{1}{2}Ct^2, \tag{3.15}$$

and thus it can then be seen that if there exists $t^* > 0$ such that

$$t^* - 1 = \frac{t^{*2}[(2/\Omega + 1) + 2t^*/\Omega]}{2[\Omega^2/\Gamma - t^* - t^{*2}/\Omega]} > 0, \tag{3.16}$$

then $X_Z(Z^*, t^*) < 0$. Conditions on Ω and Γ then follow from the requirement that the cubic equation

$$t^{*3}+\tfrac{3}{4}\Omega t^{*2}-\tfrac{1}{2}(\Omega+\Omega^3/\Gamma)\,t^*+\tfrac{1}{2}\Omega^3/\Gamma = 0 \tag{3.17}$$

has three real roots, one of which must be greater than unity. Unfortunately, this does not yield a simple relationship between Γ and Ω. However, by introducing new parameters

$$\alpha = 4/\Omega, \quad \beta = 4\Omega/\Gamma, \tag{3.18}$$

the sufficient condition for multi-valued solutions to arise in the original equations is that $0 < \alpha < \alpha_1$, where $\alpha_1(\beta)$ is the first positive zero of

$$[1+\tfrac{2}{3}(\alpha+\beta)]^3-(\alpha+1)^2(\beta+1)^2 = 0. \tag{3.19}$$

In addition it can be shown that the time t_c at which multi-valued solutions first occur satisfies

$$t_c < \frac{[1+\frac{2}{3}(\alpha+\beta)]^{\frac{1}{2}}-1}{\alpha}. \tag{3.20}$$

Solving (3.19) numerically, the condition for overturning is found to take the form $\Gamma < \Gamma^*(\Omega)$. In the next section this sufficient condition, obtained by bounding arguments, is compared with the condition suggested by a full numerical investigation. It is to be expected that the former condition is extremely restrictive compared to the latter, bearing in mind the coarseness of the bounds taken at various points of the analysis.

Before moving on to describe the numerical results, there is one interesting consequence of the conditions obtained by the functional analysis arguments provided here. Modifying the analysis slightly for a forward-facing step-like transition, it is found that even when a steadily translating solution exists (i.e. a fully dispersed relaxation shock exists), it is still possible for overturning to occur if the initial wave slope is very high; the smaller the relaxation parameters, the steeper the initial slope required to produce multi-valued solutions. More details are given in Hammerton (1990). Introducing a dimensional steepness parameter S_0, defined by

$$S_0 = -\frac{1}{U_0}\inf\left(\frac{\mathrm{d}\hat{u}}{\mathrm{d}\hat{x}}\right)_{\hat{t}=0}, \tag{3.21}$$

then the condition for overturning becomes $S_0 > S^*(U_0)$. $S^*(U_0)$ is given implicitly by

$$S^*(U_0) = \frac{2}{\alpha_0 \tau \alpha_1}, \quad U_0 = \frac{\beta(\Delta a)}{\gamma+1}, \tag{3.22}$$

where $\alpha_1(\beta)$ is defined as before, by (3.19). The curve $S^*(U_0)$ is plotted in the steepness–amplitude plane in figure 1. Whether a fully dispersed shock solution exists depends only on the disturbance amplitude. The critical value $U_0 = 4\Delta a/(\gamma+1)$ is shown as the dotted line in figure 1, and it may be seen that in region A, steady travelling wave solutions exist, but wave overturning still occurs due to the steepness of the initial profile. For those values of U_0 and S_0 where travelling wave solutions exist, but the wave still becomes multi-valued, the long-range evolution of the disturbance is not immediately clear. A sub-shock embedded in the relaxation shock must arise at finite range. The most likely scenario is that the relaxation-dominated region subsequently evolves such that the magnitude of the sub-shock decreases and the fully dispersed travelling wave solution is finally attained. Whether this is indeed the case could be determined by numerical solution. However, the sub-shock, narrow compared with the relaxation shock, must be resolved accurately, making a numerical analysis computationally very expensive.

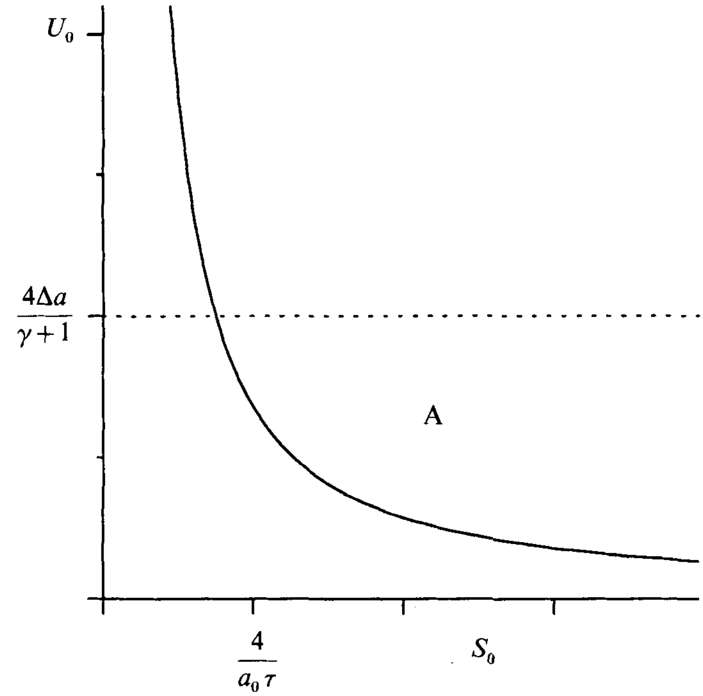

FIGURE 1. Classification of the different types of evolution of a step-like disturbance, depending on the amplitude U_0 and initial wave steepness S_0. If $S_0 > S^*(U_0)$ then multi-valued solutions will certainly occur. If $U_0 < U_s = 4\Delta a/(\gamma+1)$ then fully dispersed travelling wave solutions exist. Region A corresponds to conditions for which fully dispersed travelling wave solutions exist, but where wave overturning will still occur.

The result that triple-valued solutions can occur, even when fully dispersed solutions exist, is consistent with the work of Naumkin & Shishmarev (1982, 1983) on the phenomenon of wave breaking in Whitham's equation,

$$u_t + uu_x + \int_{-\infty}^{\infty} k(x-y)\, u_y(y,t)\,\mathrm{d}y = 0. \tag{3.23}$$

This equation describes the propagation of nonlinear waves in strongly dispersive media and was proposed by Whitham (1967) as a simple model to describe wave peaking and the breaking of waves. For sufficiently smooth kernels, $k(x)$, it can be shown that the nonlinear term prevails over the integral term so that a sufficiently steep wave will break in finite time. However, a strong enough singularity in the kernel can lead to the dominance of the integral term over the nonlinearity and hence wave overturning may be prevented. Naumkin & Shishmarev give various theorems concerning how the nature of the singularity of the kernel affects wave breaking. With suitable choice of kernel, namely

$$k(x) = -\frac{\Gamma}{\Omega^2}[1 - H(x)]\,\mathrm{e}^{x/\Omega}, \tag{3.24}$$

the relaxation equation governing the propagation of a smooth step transition can be written in the Whitham form. This kernel falls into a class for which Naumkin & Shishmarev (1982) conclude that overturning will occur in finite time as long as the initial wave slope is steep enough, exactly the conclusion of the above analysis.

Unfortunately, in the English translation of the original Russian paper, no proof of this theorem is given and no information is given on just how steep the initial gradients must be and how this criterion depends on the particular kernel. Thus no quantitative comparison can be made at present between estimates obtained in the present paper and this earlier work.

Turning to the case of periodic disturbances, the governing equation (3.2) for a relaxing medium cannot be written in the Whitham form, and thus the results of Naumkin & Shishmarev do not seem directly applicable to the main problem addressed in this paper.

4. Numerical results

For a periodic disturbance, the constraint of periodicity results in the integrated form of the governing (2.3) becoming

$$u_t + uu_\theta = -\frac{\Gamma}{\Omega}u_\theta - \frac{\Gamma}{\Omega^2}e^{\theta/\Omega}\left[J(\theta,t) - \frac{J(2\pi,t)}{1-e^{-2\pi/\Omega}}\right], \tag{4.1}$$

where

$$J(\theta,t) = \int_0^\theta u_{\theta'}\,e^{-\theta'/\Omega}\,d\theta',$$

and where the dimensionless parameters Ω and $\Gamma = \Omega K$ are defined by (2.2). The diffusivity term has been dropped in order to establish when relaxation effects alone are sufficient to fully describe the wave evolution. Attention is restricted to an initially sinusoidal disturbance. With the governing equation written in the form (4.1), the intrinsic coordinate scheme described in Part 1 can immediately be utilized, with

$$h(s,t) = -\frac{\Gamma}{\Omega^2}\left\{\Omega\tan\psi - e^{X/\Omega}\left[\hat{J}(s,t) - \frac{1}{1-e^{-2\pi/\Omega}}\hat{J}(S,t)\right]\right\}, \tag{4.2}$$

where

$$\hat{J}(s,t) = \int_0^s \sin\psi\,e^{-X(s')/\Omega}\,ds', \quad X(s,t) = \int_0^s \cos\psi\,ds'.$$

It must be noted that if the wave were to overturn, then $\cos\psi(s)$ would be zero for some s and hence $h(s,t)$ would become infinite at this point. However it is clear that $h\cos\psi$ remains finite and hence the intrinsic coordinate wave equation given by Part 1, equation (3.10b), rather than equation (3.10a), should be used for this particular governing equation. The set of equations (Part 1: (3.10b)–(3.12)) with initial conditions (Part 1: (3.13)–(3.14)) were then integrated using the method described in Part 1.

With $h(s,t)$ given by (4.2), the resulting equations are much more complicated than those described in Part 1. Thus additional accuracy checks were performed by comparing results from the intrinsic coordinate formulation with solutions obtained directly from the physical equation (4.1) using a pseudo-spectral scheme. Choosing relaxation parameter values $\Gamma = 0.5$ and $\Omega = 0.75$, the intrinsic coordinate formulation predicted wave overturning at $t = 1.25$, while the direct calculation began to break down at $t \approx 1.2$ with rapid growth in the highest spectral components. The results obtained by the two methods were compared prior to this breakdown. At $t = 0.5$ and 1.0 the differences, relative to the wave amplitude, were found to be less than 0.25%.

For the most part we are only interested in the conditions for which overturning occurs. Thus attention is centred on the magnitude of the maximum slope, $\psi_{max}(t)$, as time changes. Assuming that nonlinear effects initially dominate, the magnitude of negative gradients will start to increase (wave steepening) and if $|\psi_{max}(t)|$ exceeds $\frac{1}{2}\pi$,

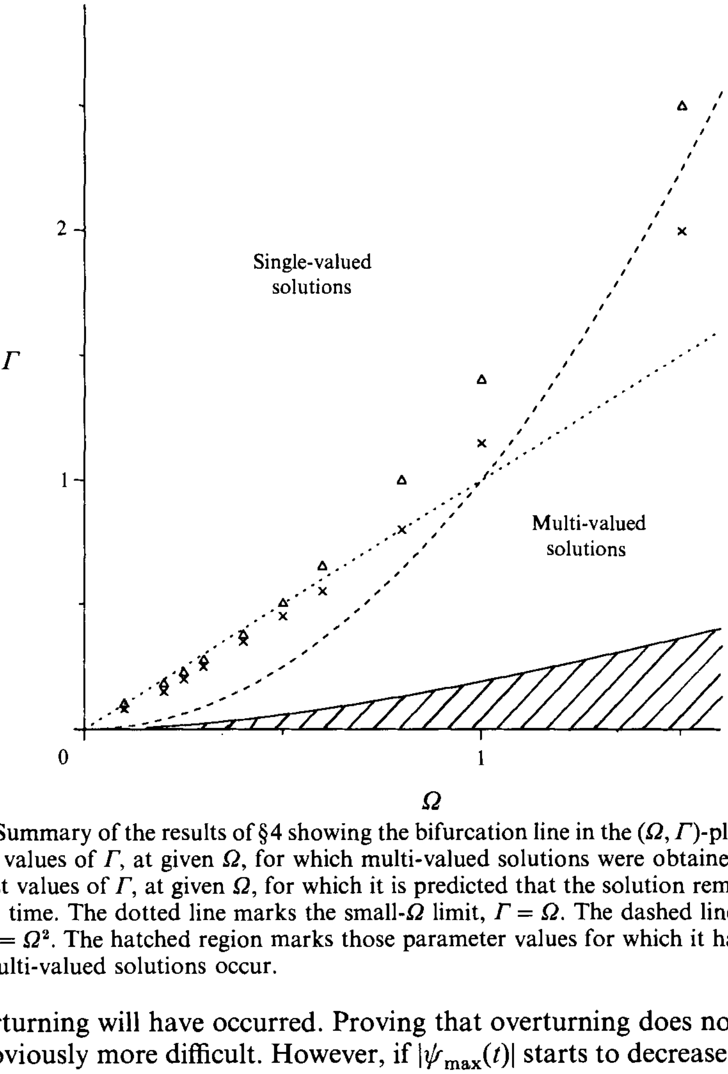

FIGURE 2. Summary of the results of §4 showing the bifurcation line in the (Ω, Γ)-plane. Crosses mark the largest values of Γ, at given Ω, for which multi-valued solutions were obtained. Triangles mark the smallest values of Γ, at given Ω, for which it is predicted that the solution remains single-valued at all finite time. The dotted line marks the small-Ω limit, $\Gamma = \Omega$. The dashed line marks the large-Ω limit, $\Gamma = \Omega^2$. The hatched region marks those parameter values for which it has been proved, in §3, that multi-valued solutions occur.

then overturning will have occurred. Proving that overturning does not occur in finite time is obviously more difficult. However, if $|\psi_{max}(t)|$ starts to decrease, corresponding to relaxation effects outweighing nonlinear steepening, it seems reasonable to assume that the maximum forward-facing slope will continue to decrease in magnitude. In the numerical investigation, it was asserted that if the maximum wave slope began to decrease, or obviously tended to a limit well below $\frac{1}{2}\pi$, then the wave would remain single-valued for all finite times. Use of this criterion evidently leaves some cases where the long-time behaviour is not fully determined, but this appears to be unavoidable.

Without detailed investigation of the possibility of wave overturning, it is readily seen that for $\Gamma = 0$ the inviscid Burgers equation is obtained and wave overturning will occur whatever the value of Ω, while for $\Omega = 0$ (and $\Gamma \neq 0$) the full Burgers equation is obtained and the solution remains single-valued at all times. In addition it was proved in §3 that overturning occurs for non-zero values of the relaxation parameters. Thus some bifurcation line must exist dividing the (Ω, Γ)-plane into a region for which multi-valued solutions appear and a region for which the wave solution always remains single-valued. Using the intrinsic coordinate method, the position of this bifurcation line in parameter space can be determined. The results obtained are summarized in figure 2. Various values of Ω were chosen, then Γ was varied to obtain the largest value for which overturning occurred and the smallest value for which the presence of

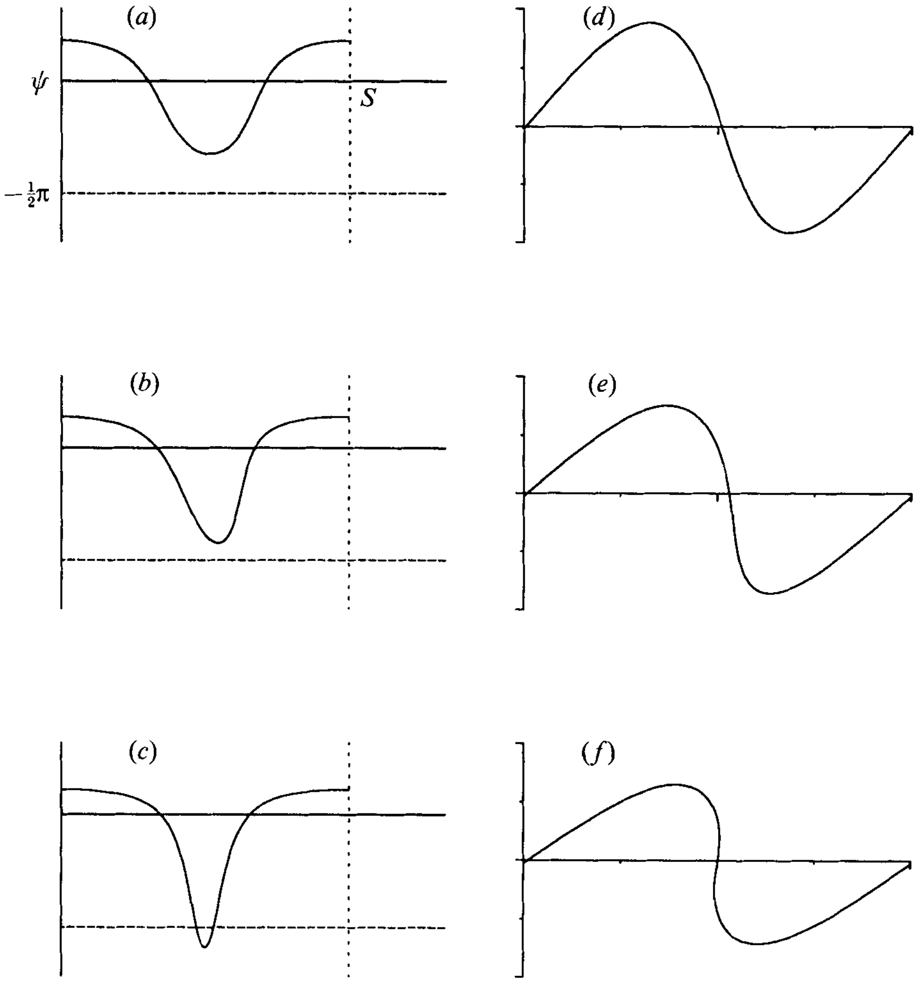

FIGURE 3. Evolution of a sinusoidal disturbance as it propagates through a relaxing medium with $\Omega = 0.5$ and $\Gamma = 0.25$. These conditions result in the appearance of multi-valued solutions. (*a–c*) Intrinsic wave functions and (*d–f*) the corresponding physical waveforms. (*a*, *d*) $t = 0.5$; (*b*, *e*) $t = 1.0$; (*c*, *f*) $t = 1.5$.

overturning could be discounted. It can be seen that the position of the bifurcation line is determined to a reasonable degree of accuracy. The problems in reducing the degree of uncertainty are discussed later in this section.

In figures 3–5 the typical evolution of the intrinsic wave function and of the corresponding physical waveform is shown for different parameter values. In each case Ω is set to 0.5 and then various values of Γ are taken to illustrate the wave behaviour at different positions in the parameter space relative to the bifurcation line. Figures 3 and 4 show the behaviour of the solution for parameter values well away from the bifurcation line. First it may be noted that in these cases the intrinsic wave function remains a fairly smooth function of arclength, corresponding to the absence of sharp changes in gradient of the real waveform. Second, it can be seen that for these parameter values there can be no element of doubt as to whether wave overturning has, or has not, occurred. In figure 3, with $\Gamma = 0.25$, overturning is readily identified, with the value of the intrinsic wave function passing below $-\frac{1}{2}\pi$ over an appreciable spatial

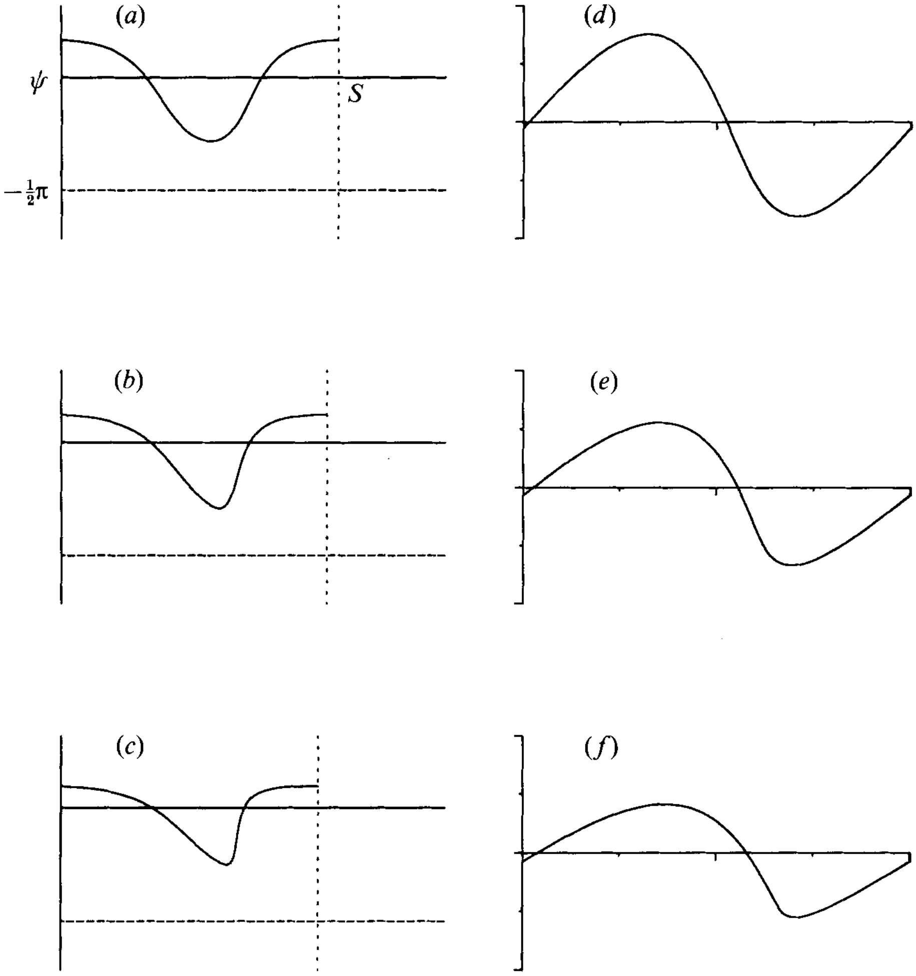

FIGURE 4. Evolution of a sinusoidal disturbance as it propagates through a relaxing medium with $\Omega = 0.5$ and $\Gamma = 0.6$. These relaxation conditions ensure that the solution remains single-valued. (*a–c*) Intrinsic wave functions and (*d–f*) the corresponding physical waveforms. (*a, d*) $t = 0.5$; (*b, e*) $t = 1.0$; (*c, f*) $t = 1.5$.

range. In figure 4, where $\Gamma = 0.6$, the maximum wave slope is initially seen to increase, but then steadily decrease and here it is clear that wave overturning will never occur. However, close to the dividing line, the situation becomes less clear-cut. This is demonstrated in figure 5 for $\Gamma = 0.4$. In this case, near-discontinuities appear in the intrinsic solution, with the largest slopes being confined to an extremely small region. Most importantly, these sharp changes in ψ occur in the immediate neighbourhood of the maximum value of $|\psi|$. Since we are interested in whether the maximum slope ever exceeds $\frac{1}{2}\pi$, accuracy in evaluating ψ in this region of rapid change is vital. Care was indeed taken to vary the temporal and spatial resolution to ensure that such variation in ψ is genuinely prescribed by the governing equation and is not a numerical invention. The sharp changes in wave slope calculated correspond to the appearance of a near 'kink' in the physical waveform, which is similar to that at the head of a travelling shock at critical wave amplitude. While figures 3(*f*) and 5(*f*) do not represent realizable solutions of the full governing equations, they give a good idea of

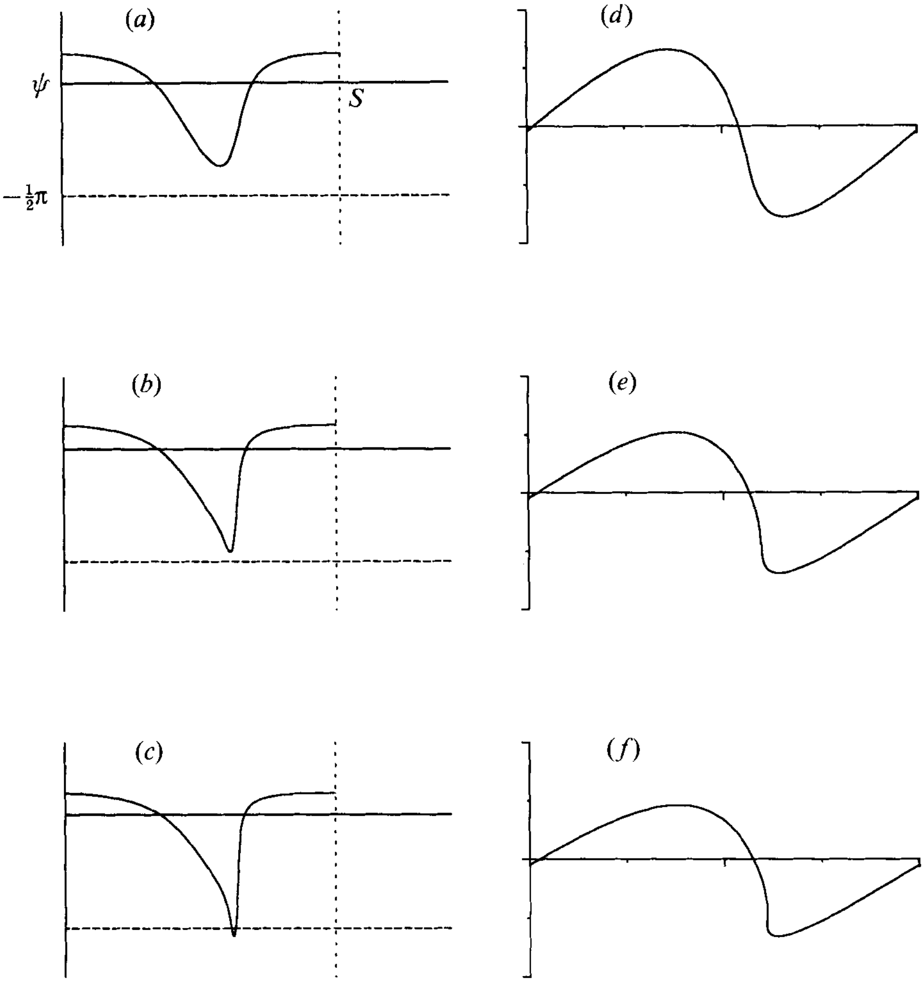

FIGURE 5. Evolution of a sinusoidal disturbance as it propagates through a relaxing medium with $\Omega = 0.5$ and $\Gamma = 0.4$. (*a–c*) Intrinsic wave functions and (*d–f*) the corresponding physical waveforms. (*a*, *d*) $t = 0.5$; (*b*, *e*) $t = 1.0$; (*c*, *f*) $t = 1.5$. Very localized overturning of the wave is seen to occur. This corresponds to parameter values close to the bifurcation line.

the overall wave shape, and where a sub-shock must be located, without requiring the very fine resolution needed to compute directly the physical waveform with embedded sub-shock.

The behaviour of the solution close to the bifurcation line, as discussed above, is one reason why the bifurcation line cannot be determined exactly. In the large-Ω limit, a different limitation is placed on the accuracy in determining the bifurcation line. Letting $\Omega \to \infty$, with $\alpha = \Gamma/\Omega^2$ held fixed, the Varley–Rogers equation is obtained (Part 1: (2.4)). Solving this equation, a frozen waveform arises at large time. For an initially sinusoidal wave of unit amplitude the frozen wave will have overturned if $\alpha < 1$, while if $\alpha > 1$ the ultimate solution is single-valued. Hence $\Gamma = \Omega^2$ is the bifurcation line in this limit, and this is marked on figure 2 as the dashed line. Close to this line, the maximum wave slope increases very slowly to its frozen value, and hence accurate prediction of the position of the bifurcation line by intrinsic coordinates is difficult. However, the numerical results obtained do seem to be consistent with the Varley–Rogers limiting case.

In the limit $\Omega \to 0$, $\Gamma = O(\Omega)$, the results obtained here can be compared with the asymptotic results of Crighton & Scott (1979). In this limit the relaxing shock is narrow, with its amplitude determined by the lossless outer solution. Analysis of the shock region suggests that if $\Gamma < \Omega$, then the solution becomes multi-valued. The bifurcation line $\Gamma = \Omega$ is plotted in figure 2. In the light of the present analysis, this condition can be modified slightly. In §3 it was proved that the maximum wave amplitude decreases with time for non-zero relaxation. This suggests that the bifurcation line should in fact be $\Gamma/\Omega = 1 - a\Omega + O(\Omega^2)$, where a is positive. That the calculated bifurcation line does in fact lie below $\Gamma/\Omega = 1$, in the small-Ω limit, is clearly shown in figure 2.

In summary, close to the bifurcation line, overturning is very much a local effect, but its occurrence depends on the global wave shape. Further away from the bifurcation line, if overturning does occur, it is less local and more akin to the standard nonlinear overturning in the absence of dispersion and attenuation.

5. Conclusion

In this paper, we have looked in detail at the role of relaxation effects in determining wave profiles. It has been shown that in some circumstances, a particular relaxation effect, though apparently the dominant mechanism on simple scaling arguments, is insufficient on its own to fully describe the nonlinear wave propagation. For a sinusoidal disturbance, two non-dimensional parameters enter the propagation problem. By analysing the phenomenon of wave overturning for such a problem, a condition on these parameters is obtained dictating whether other physical mechanisms must eventually become significant. In §3 sufficient conditions for multi-valued conditions to occur are determined by functional analysis arguments, while in §4 a more precise criterion for overturning is obtained using numerical results. The two sets of results are compared in figure 2, with the hatched region marking that part of parameter space for which overturning is rigorously predicted by the functional analysis arguments. As anticipated, the sufficiency conditions of §3 are extremely restrictive. Possibly the analysis of §3 should be interpreted more as a proof that under certain conditions, purely relaxing media cannot support single-valued solutions, rather than as a calculation of these conditions.

In figure 6, the estimated position of the bifurcation line is plotted in the frequency–amplitude plane. This estimate corresponds to taking the curve approximately bisecting the extreme values plotted in figure 2. Well above the bifurcation line, the relaxation effects are essentially negligible, only affecting the overall phase velocity of the signal. Wave overturning occurs at virtually the same point as if the relaxation mechanism were entirely ignored, and so other mechanisms must be included. Well below the bifurcation line, the relaxation manifests itself as an additional diffusion term with coefficient equal to Γ, and it is for this reason that physical mechanisms such as rotational energy effects can be included as an additional 'bulk' term in the coefficient of diffusivity. Only in the neighbourhood of the bifurcation line must relaxation effects be considered in detail.

At this point comparison may be made with other work on wave breaking in relaxing media. We have already looked at the small-Ω asymptotic analysis of Crighton & Scott (1979); a more general analysis of propagation through relaxing media is presented in a series of papers by Johannesen and co-workers. Johannesen & Scott (1978), address a problem similar to that examined in the present work, with the added complexity of non-planar geometries. The flow due to a plane or spherical piston oscillating

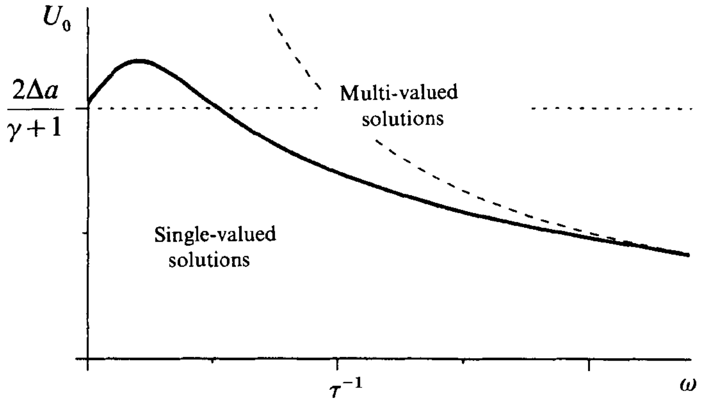

FIGURE 6. Estimated position of the bifurcation line for a harmonic disturbance, in terms of the signal amplitude (U_0) and the signal frequency (ω). The dashed lines show the large- and small-ω asymptotic limits.

harmonically is considered and conditions for wave breaking are obtained. However, the analysis is concerned only with the start-up problem; that is, the piston is set in motion at $t = 0$. The investigation is then limited to a local analysis of the head of the disturbance which propagates into the quiescent medium at the frozen sound speed. In the plane-wave case, the results obtained are identical to the high signal frequency (large-Ω) results of the present paper. This is not a complete surprise since the analysis away from the piston is restricted to the neighbourhood of the frozen-sound-speed characteristic where only high-frequency components can be present at any distance from the piston. In Southern & Johannesen (1980) numerical solutions are provided for the nonlinear propagation of plane waves with relaxation effects included. These solutions are obtained by numerically integrating the system of three gasdynamic equations written in characteristic form. The start-up problem is considered and thus two sets of characteristics propagate into the quiescent medium. Results are provided for several wavelengths behind the head of the disturbance, where transient effects are assumed to be negligible. However, these results cannot be directly compared with the predicted wave profiles of the current paper, because Southern & Johannesen compute only cases where the energy in the relaxing mode is high, with the result that the model equation considered in the present paper becomes invalid. Qualitatively the waveforms are similar, with the appearance of significant asymmetry followed by a kink close to the wave trough. Beyond this point, where our intrinsic coordinate results predict wave overturning, Southern & Johannesen see the appearance of a local oscillation in the wave trough. This difference in wave shape is most likely to be due to a different relaxation effect being modelled, but may be associated with the transient effect of start-up. Any attempt to solve the exact problem considered in the present paper using characteristics stumbles on the enforced condition of periodicity.

Finally, the correct physical interpretation of the results contained herein must be reiterated. Inclusion of relaxation with no diffusion whatsoever is not physically realistic, since those molecular vibration modes excluded from the relaxation terms due to their almost instantaneous response act as bulk viscosity terms. However, identification of wave overturning in a model equation containing just one relaxing mode points to another physical effect, with an associated lengthscale, becoming significant. Only by an analysis such as that presented here can all the relevant physical processes be identified, together with the associated scalings entering the fine structure of the wave.

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